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On the transport between condensed phases

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Abstract

An exactly solvable model for the particle transport between two reservoirs of non-interacting Bose particles through a microscopic tunnelling junction is studied. The analysis covers the whole range of temperatures and densities of the reservoirs, and the particle and energy flows between the reservoirs are calculated in closed form. In particular, in the case, when Bose–Einstein condensation is present, their dependence on the phase difference of the condensates is established.

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1. Introduction

A very simple, exactly soluble, model describing a tunnelling junction of two Bose reservoirs in different equilibrium states is considered. We show the existence, and construct, the non-equilibrium stationary state attained at large time starting from the constrained equilibrium state of the uncoupled reservoirs, when allowing tunnelling through a microscopic channel. Thereby, we are interested in the case, not considered before, when the reservoirs are allowed to be in Bose-condensed phases. The most widely known example of physical phenomenon of the transport between reservoirs in condensed phases is provided by the Josephson current between two superconductors. The tunnelling of bosons in the model we study, while far from being a microscopic model for a Josephson junction, can be viewed as a caricature of the pair tunnelling between two BCS states.

We consider two lattice-free Bose gases on $\mathbb{Z}^3$ with a tunnelling junction through the $x = 0$ sites. More precisely, the Hamiltonian in the symmetric Fock space $\mathcal{F} = \mathcal{F}(\mathcal{H})$ is taken as

$$H_\lambda = -\frac{1}{2}\sum_{x,y\in\mathbb{Z}^3} t_{x,y}(a_x^* a_y + b_x^* b_y) + \lambda(a_0^* b_0 + b_0^* a_0), \tag{1.1}$$

where $t_{x,y} = \delta_{|x-y|,1} - 6\delta_{x,y}$, and $a_x^\sharp, b_x^\sharp$ are the creation–annihilation operators for bosons in the first and second reservoir, respectively. Here, $\mathcal{H} = l^2(\mathbb{Z}^3) \bigoplus l^2(\mathbb{Z}^3)$ is the space of one-boson states. For our purpose, the coefficients can be taken arbitrary such that $t_{x,y} = t(x-y) \geqslant 0$, if $x \neq y, t_{x,y} = 0$, if $|x-y| > R$, and $\sum_y t_{x,y} = 0$; our special choice corresponds to a discrete approximation of the usual kinetic energy, i.e., $-\frac{1}{2}\Delta$.

Actually, we need the evolution of the two infinitely extended reservoirs, prepared initially in two different equilibrium states at given temperatures and densities, after switching on the tunnelling coupling. The Fock space Hamiltonian (1.1) is, however, not adequate for such a setting. In order to describe the dynamics of the infinite system, we use the algebraic formulation of quantum statistical mechanics [1, 2]: we choose an appropriate algebra of local observables and view (1.1) as generating the time evolution automorphism on it. Such an approach to studying non-equilibrium stationary states has been extensively used in the last years, mostly for reservoirs of independent Fermions (see [2] and references therein).

Already considering Bose, instead of Fermi, statistics is a non-trivial technical task, even when no condensation is present [3]. Condensation implies non-uniqueness of the equilibrium state of the reservoir and hence requires a refined setting. In a recent paper, the simpler problem of the return to equilibrium of a reservoir of non-interacting Bose particles with condensate interacting with a finite quantum system has been studied [4]. When the finite quantum system is replaced by another reservoir, a new features are expected due to the fact that an arbitrarily large number of particles are involved in the interaction; in particular, instead of an equilibrium state (which, in fact, no longer exists [5]), the system reaches a stationary state of the coupled system in which non-zero flows of particles and energy are present.

The choice of a coupling *bilinear* in $a_x^\sharp, b_x^\sharp$ between reservoirs, while physically reasonable, is the essential simplifying feature of our model. Under the bilinearity assumption, the Hamiltonian (1.1) is the second quantization of its restriction to the one-particle subspace $\mathcal{H} \subset \mathcal{F}(\mathcal{H})$, i.e., of a one-particle Hamiltonian h_λ. The evolution of local observables which are linear combinations of creation/annihilation operator, $\sum_x \xi_x a_x^\sharp + \eta_x b_x^\sharp$, is given by evolving the vector $(\xi, \eta) \in \mathcal{H}$ according to h_λ. Under these condition, not only the initial state, which is taken as a product of equilibrium states of the two (uncoupled) gases, but also the time-evolved state is quasi-free states. The existence of the $t \to \infty$ limiting state and its explicit form as a quasi-free state is thus reduced to finding the Möller operators of the pair (h_0, h_λ) of one-particle Hamiltonians. Thereby, the particle and energy currents between the two reservoirs in the stationary state, in particular the contribution of the condensate, is obtained in closed form. The other simplifying assumptions of the model can be relaxed and the same kind of results obtained under somewhat more general conditions (e.g. for several reservoirs modelled as continuous or lattice Bose gases with different dispersion laws or tunnelling through an arbitrary finite number of sites, etc) at the price of more complicated formulae.

The results we obtain for this oversimplified model may, however, give a hint on what is to be expected regarding the contribution of the condensate in more general cases (e.g. polynomial coupling of higher degree in the creation–annihilation operators between the reservoirs).

2. The dynamics on the algebra of the canonical commutation relations

It will be convenient to identify the one-particle Hilbert space $\mathcal{H} = l^2(\mathbb{Z}^3) \bigoplus l^2(\mathbb{Z}^3)$ with the space $l^2(\mathbb{Z}^3, \mathbb{C}^2)$ of square summable sequences

$$f = \left\{ \begin{pmatrix} \xi_x \\ \eta_x \end{pmatrix}; x \in \mathbb{Z}^3 \right\}$$

of two-dimensional vectors, where x labels the lattice sites and the upper (lower) component refers to the first (second) reservoir. Let $\pi_0 : \mathcal{H} \to \mathbb{C}^2$ be the restriction of f to the $x = 0$ site,

$$\pi_0 f = \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}.$$

With this identification, the one-particle Hamiltonian writes

$$\begin{aligned} h_\lambda &= h_0 + \lambda v \\ &= -\tfrac{1}{2}\Delta I + \lambda \pi_0^* \sigma_1 \pi_0, \end{aligned} \tag{2.1}$$

where Δ denotes the lattice Laplace operator, I is the unit operator in $\mathbb{C}^2$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the first Pauli matrix.

The kinematical C^*-algebra of the model is the canonical commutation relation algebra $\mathcal{W}(\mathcal{D})$ over the invariant subspace of h_λ, $\mathcal{D} = l^1(\mathbb{Z}^3, \mathbb{C}^2) \subset \mathcal{H}$ (the reason for this choice of $\mathcal{D}$ will become clear in section 3). $\mathcal{W}(\mathcal{D})$ is generated by the Weyl operators $\{W(f);\ f \in \mathcal{D}\}$, satisfying

$$W(f)W(g) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(f,g)} W(f+g). \tag{2.2}$$

The time evolution for the coupled reservoirs is the group of Bogoliubov automorphisms on $\mathcal{W}(\mathcal{D})$ defined by its action on $W(f)$:

$$\tau_t^\lambda(W(f)) = W(\mathrm{e}^{\mathrm{i}h_\lambda t} f). \tag{2.3}$$

In view of the canonical commutation relations (2.2), equation (2.3) is sufficient to uniquely define the action of τ_t^λ on all elements of $\mathcal{W}(\mathcal{D})$. Likewise, the evolution of the uncoupled reservoirs is τ_t^0 given by equation (2.3) with $\lambda = 0$.

Suppose that, at time $t = 0$, when the coupling between reservoirs is switched on, the system was in an (constrained) equilibrium state of the uncoupled reservoirs, ω_0 on $\mathcal{W}(\mathcal{D})$. In particular, ω_0 is invariant with respect to the free evolution τ_t^0 (i.e. for any $A \in \mathcal{W}(\mathcal{D})$, the expectation $\omega_0(\tau_t^0(A))$ is independent of t). The expectations of the observables A at time $t > 0$, given by $\omega_0\big(\tau_t^\lambda(A)\big) =: \omega_t(A)$, define the evolved state ω_t. In view of the τ_t^0-invariance of ω_0,

$$\omega_t(A) = \omega_0\big(\tau_t^\lambda(A)\big) = \omega_0\big(\tau_{-t}^0 \circ \tau_t^\lambda(A)\big), \tag{2.4}$$

therefore, its large-t asymptotics is controlled by the scattering for the pair $\big(\tau_t^\lambda, \tau_t^0\big)$ of evolutions. As for the generators $W(f)$ of $\mathcal{W}(\mathcal{D})$ equation (2.4) reads $\tau_{-t}^0 \tau_t^\lambda(W(f)) = W(\mathrm{e}^{-\mathrm{i}h_0 t}\,\mathrm{e}^{\mathrm{i}h_\lambda t} f)$, the limit is in fact controlled by the scattering of the two one-particle unitary evolutions $(\mathrm{e}^{\mathrm{i}h_\lambda t}, \mathrm{e}^{\mathrm{i}h_0 t})$.

In this section, we perform the spectral analysis of the Hamiltonians h_0 and h_λ and calculate the relevant Möller operators.

The unperturbed Hamiltonian h_0 is readily analysed by Fourier transform, which is the unitary $u : \mathcal{H} \to \hat{\mathcal{H}} := L_2([0, 2\pi)^3, \mathbb{C}^2)$:

$$(uf)(k) = (2\pi)^{-3/2} \sum_{x \in \mathbb{Z}^3} \mathrm{e}^{\mathrm{i}kx} f_x, \qquad f \in \mathcal{H} \tag{2.5}$$

$$(u^* \hat{f})_x = (2\pi)^{-3/2} \int_{[0,2\pi)^3} \mathrm{e}^{-\mathrm{i}kx} \hat{f}(k)\,\mathrm{d}k, \qquad \hat{f} \in \hat{\mathcal{H}}, \tag{2.6}$$

showing its unitary equivalence with the multiplication by $\omega(k) = \sum_{\alpha=1}^{3} 2\sin^2(k_\alpha/2)$:

$$(uh_0u^*\hat{f})(k) = \omega(k)\hat{f}(k). \tag{2.7}$$

Hence, the spectrum of h_0 equals the range of ω, i.e., the real interval $[0, 6]$, and the generalized eigenfunctions corresponding to $z = e \in [0, 6]$ are given by

$$\psi^0_{p,1} = u^*\begin{pmatrix}\delta_p\\0\end{pmatrix}, \qquad \psi^0_{p,2} = u^*\begin{pmatrix}0\\\delta_p\end{pmatrix},$$

where $\delta_p(k) = \delta(k-p)$ is the Dirac function and p runs over $\{p \in [0, 2\pi)^3 : \omega(p) = e\}$.

The resolvent of h_λ, $R_\lambda(z) = (h_\lambda - z)^{-1}$ is calculated by solving the equation

$$(h_0 - z)f + \lambda\pi_0^*\sigma_1\pi_0 f = g. \tag{2.8}$$

This implies an equation for $f_0 = \pi_0 f \in \mathbb{C}^2$:

$$(I + \lambda\pi_0(h_0 - z)^{-1}\pi_0^*\sigma_1)f_0 = \pi_0(h_0 - z)^{-1}g. \tag{2.9}$$

Define $G(z) = \{g_x(z); x \in \mathbb{Z}^3\}$, where

$$g_x(z) = (2\pi)^{-3}\int_{[0,2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}^3k}{\omega(k) - z} \tag{2.10}$$

is the matrix element $\left[\left(-\frac{1}{2}\Delta - z\right)^{-1}\right]_{x,0}$. Then,

$$\pi_0(h_0 - z)^{-1}\pi_0^* = g_0(z) \times I \tag{2.11}$$

and we have

$$\det(I + \lambda\pi_0(h_0 - z)^{-1}\pi_0^*\sigma_1) = 1 - \lambda^2 g_0(z)^2. \tag{2.12}$$

Remark 2.1. The function $g_0(z)$ is analytic in $\mathbb{C}\backslash[0, 6]$ and has continuous boundary values at the cut (see, e.g., [6]). We denote

$$\gamma(k) := \lambda g_0(\omega(k) + \mathrm{i}0). \tag{2.13}$$

In the model on $\mathbb{Z}^d$, $\gamma(0)$ is real and finite for $d \geqslant 3$, in particular, in our $d = 3$ case, $0 < \gamma(0) < \infty$. Also, $\operatorname{Im}\gamma(k) \neq 0$ for $\omega(k) \in (0, 6)$. Finally, as $\omega(k) - 6 = -\omega(k')$, where $k'_\alpha = \pi - k_\alpha$, we have that $g_x(6 - z) = -\mathrm{e}^{\mathrm{i}\pi\sum_\alpha x_\alpha} g_x(z)$.

As a consequence, if $\gamma(0) > 1$, the determinant has exactly two simple zeros at two real points $e_0 < 0$ and $6 - e_0$. Thereby, if $\tilde{\phi}_0 \in \mathbb{C}^2$ is in the kernel of the matrix $I + \lambda g_0(e_0)\sigma_1$, then $\phi^{(e_0)} = -\lambda(h_0 - e_0)^{-1}\pi_0^*\sigma_1\tilde{\phi}_0$ satisfies $\pi_0\phi = \tilde{\phi}_0$, hence

$$(h_0 - e_0)\phi + \lambda\pi_0^*\sigma_1\pi_0\phi = 0.$$

Hence, e_0, $6 - e_0$ are simple eigenvalues of h_λ and

$$\phi^{(e_0)} = NG(e_0)\begin{pmatrix}-\lambda g_0(e_0)\\1\end{pmatrix}, \qquad \phi^{(6-e_0)} = NG(6 - e_0)\begin{pmatrix}-\lambda g_0(6 - e_0)\\1\end{pmatrix}$$

are the associated normalized eigenvectors. On the other hand, if $\gamma(0) \leqslant 1$, the determinant does not vanish for $z \in \mathbb{C}\backslash[0, 6]$, implying that there is no spectrum outside $[0, 6]$.

If $I + \lambda g_0(z)\sigma_1$ is invertible, z belongs to the resolvent set of h_λ and, replacing $\pi_0 f$ in equation (2.8) with the solution f_0 of equation (2.9), one obtains the explicit form of the resolvent:

$$\begin{aligned}R_\lambda(z) &:= (h_\lambda - z)^{-1}\\ &= R_0(z) - R_0(z)\lambda\pi_0^*\sigma_1(I + \lambda g_0(z)\sigma_1)^{-1}\pi_0 R_0(z).\end{aligned} \tag{2.14}$$

The strong limits

$$\text{strong} - \lim_{t\to-\infty} \mathrm{e}^{\mathrm{i}th_\lambda}\,\mathrm{e}^{-\mathrm{i}th_0} = \Omega_- \tag{2.15}$$

exist and define an isometry $\Omega_- : \mathcal{H} \to \mathcal{H}_{ac}$ (Möller operator), where $\mathcal{H}_{ac} \subset \mathcal{H}$ is the subspace of absolute continuity of h_λ [7]. Known formulae of scattering theory [7] allow us to obtain the generalized eigenfunctions of h_λ from those of h_0 as

$$\psi_{p,i} = \Omega_-\psi^0_{p,i} = -\lim_{\epsilon\searrow 0} \mathrm{i}\epsilon R_\lambda(\omega(p)+\mathrm{i}\epsilon)\psi^0_{p,i}. \tag{2.16}$$

As $-\mathrm{i}\epsilon R_0(\omega(p)+\mathrm{i}\epsilon)\psi^0_{p,i} = \psi^0_{p,i}$, we obtain

$$\begin{aligned}\psi^\lambda_{p,1} &= \psi^0_{p,1} - \frac{\lambda G(\omega(p)+\mathrm{i}0)}{1-\gamma(p)^2}\begin{pmatrix}-\gamma(p)\\ 1\end{pmatrix}\\ \psi^\lambda_{p,2} &= \sigma_1\psi^\lambda_{p,1}.\end{aligned} \tag{2.17}$$

In Fourier transform

$$\hat{\psi}^\lambda_{p,1} = \begin{pmatrix}\delta_p\\ 0\end{pmatrix} - \frac{\lambda}{1-\gamma(p)^2}\cdot\frac{(2\pi)^{-3}}{\omega(k)-\omega(p)-\mathrm{i}0}\begin{pmatrix}-\gamma(p)\\ 1\end{pmatrix}. \tag{2.18}$$

As a consequence of the above calculations, we have that the wave operator Ω_- has the form $u\Omega_-u^* = I + K$, where I is the unit operator and K has the generalized kernel:

$$K(k,k') = \frac{\lambda}{1-\gamma(k')^2}\cdot\frac{(2\pi)^{-3}}{\omega(k)-\omega(k')-\mathrm{i}0}\begin{pmatrix}\gamma(k') & -1\\ -1 & \gamma(k')\end{pmatrix}. \tag{2.19}$$

Under our assumptions, the wave operator for the pair (h_λ, h_0) likewise exists and

$$\text{strong} - \lim_{t\to\infty} \mathrm{e}^{-\mathrm{i}th_0} P_{\mathcal{H}_{ac}}\,\mathrm{e}^{\mathrm{i}th_\lambda} = \Omega_-^*, \tag{2.20}$$

where $P_{\mathcal{H}_{ac}}$ is the orthogonal projection onto $\mathcal{H}_{ac}$.

Proposition 2.1. *Let $\gamma(0) \leqslant 1$. Then, $\forall f \in \mathcal{D}$, the following limit exists and defines an automorphism of $\mathcal{W}(\mathcal{D})$:*

$$\lim_{t\to\infty} \tau^0_{-t}\circ\tau^\lambda_t W(f) = W(\Omega^*_- f). \tag{2.21}$$

Proof. As, for $\gamma(0) \leqslant 1$, h_λ has no eigenvalues, $\mathcal{H}_{ac} = \mathcal{H}$ and the assertion follows from equation (2.20) taking into account the definition (2.3). □

3. The constrained equilibrium state and its time evolution

To any state ω on $\mathcal{W}(\mathcal{D})$ is associated its (nonlinear) generating functional $E : \mathcal{D} \to \mathbb{C}$:

$$\omega(W(f)) = E(f), \tag{3.1}$$

which satisfies (i) normalization $E(0) = 1$, (ii) unitarity $\overline{E(f)} = E(-f)$ and (iii) positivity:

$$\sum_{i,j=1}^{n} z_i E(f_i - f_j)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(f_i,f_j)}\bar{z}_j \geqslant 0, \qquad \forall n, \forall z_i \in \mathbb{C}, \qquad f_i \in \mathcal{D} \quad (i = 1,\ldots,n).$$

Conversely, any E with these properties defines a unique state by equation (3.1). Therefore, in describing the initial and evolved states of our model, it will be sufficient to specify the corresponding generating functionals.

The initial state will be taken as a product of canonical equilibrium states of the two reservoirs at temperatures β_i and densities ρ_i, $i = 1, 2$:

$$E_0(f) = E^{(1)}_{\beta_1,\rho_1}(f_1)E^{(2)}_{\beta_2,\rho_2}(f_2). \tag{3.2}$$

We need, therefore, a short description of the canonical states of a lattice Bose reservoir. We adapt the derivation by Cannon [8] for the continuum Bose gas. Let β, ρ be fixed positive numbers and define

$$\rho_{\mathrm{cr}}(\beta) = (2\pi)^{-3} \int_{[0,2\pi)^3} \frac{1}{\mathrm{e}^{\beta\omega(k)} - 1}\, \mathrm{d}^3k.$$

For $\rho < \rho_{\mathrm{cr}}(\beta)$, the fugacity z is defined to be the unique solution of the equation

$$\rho = (2\pi)^{-3} \int_{[0,2\pi)^3} \frac{z}{\mathrm{e}^{\beta\omega(k)} - z}\, \mathrm{d}^3k,$$

while, for $\rho \geqslant \rho_{\mathrm{cr}}(\beta)$, put $z = 1$. The momentum distribution for $k \neq 0$ at the given β, ρ is proportional to

$$n(k) = \frac{z}{\mathrm{e}^{\beta\omega(k)} - z},$$

while the condensate density is given by

$$\rho_0 = \max\{0, \rho - \rho_{\mathrm{cr}}(\beta)\}.$$

Then, the generating functional of the canonical equilibrium state at β, ρ is given by the formula

$$E_{\beta,\rho}(f) = \exp\left\{-\frac{\|f\|^2}{4} - \frac{1}{2}(\hat{f}, n\hat{f})\right\} J_0\big(\sqrt{2(2\pi)^3\rho_0}|\hat{f}(0)|\big), \tag{3.3}$$

where J_0 is the Bessel function.

For $\rho \leqslant \rho_{\mathrm{cr}}(\beta)$, the canonical state defined by equation (3.3) is extremal; however, if $\rho > \rho_{\mathrm{cr}}(\beta)$, it has a non-trivial decomposition into extremal states indexed by a phase $\mathrm{e}^{\mathrm{i}\theta}$:

$$E_{\beta,\rho}(f) = (2\pi)^{-1} \int_0^{2\pi} E^{\theta}_{\beta,\rho}(f)\, \mathrm{d}\theta, \tag{3.4}$$

where

$$E^{\theta}_{\beta,\rho}(f) = \exp\left\{-\frac{\|f\|^2}{4} - \frac{1}{2}(\hat{f}, n\hat{f})\right\} \exp\{-\mathrm{i}(2\pi)^{-3/2}\sqrt{2\rho_0}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\theta} f(0))\}. \tag{3.5}$$

Thereby, $f \in l^2(\mathbb{Z}^3)$ is such that the functionals (3.5) are well defined, e.g., $f \in l^1(\mathbb{Z}^3)$ will suffice.

After this preparation, we come back to our model of two reservoirs. In view of the above discussion, a natural choice of the test function subspace is $\mathcal{D} = l^1(\mathbb{Z}^3, \mathbb{C}^2)$. Then, equation (3.2) with $E^{(i)}_{\beta_i,\rho_i}(f_i)$ arbitrary mixtures (with probability measures $\mathrm{d}\mu_i(\theta_i)$) of extremal states (3.5) at $\beta_i, \rho_i, i = 1, 2$, defines a state over $\mathcal{W}(\mathcal{D})$. Denoting

$$\tilde{n}_0 = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}, \qquad \tilde{\rho}_0(\theta_1, \theta_2) = (\sqrt{2\rho_{01}}\,\mathrm{e}^{-\mathrm{i}\theta_1} \sqrt{2\rho_{02}}\,\mathrm{e}^{-\mathrm{i}\theta_2}), \tag{3.6}$$

we have

$$E_0(f) = \int \mathrm{d}\mu_1(\theta_1)\, \mathrm{d}\mu_2(\theta_2) E_0^{\theta_1,\theta_2}(f), \tag{3.7}$$

where

$$E_0^{\theta_1,\theta_2}(f) = \exp\left\{-\frac{\|f\|^2}{4} - \frac{(\hat{f}, \tilde{n}_0\hat{f})}{2} - \frac{\mathrm{i}}{(2\pi)^{3/2}}\mathrm{Re}(\tilde{\rho}_0(\theta_1, \theta_2) \cdot \hat{f}(0))\right\}. \tag{3.8}$$

In particular, the canonical states (3.3) are obtained for $\mathrm{d}\mu_i(\theta) = (2\pi)^{-1}\,\mathrm{d}\theta$.

We are interested in the time evolution of an initial state ω_0 as defined by equation (3.7) under the coupled dynamics. The evolution (2.4) is quasi-free. As a consequence of proposition 2.1, we obtain the following convergence result, which defines the stationary state:

Proposition 3.1. *Let $\gamma(0) \leqslant 1$. Then, $\forall f \in \mathcal{D}$, the following limit exists:*

$$\lim_{t\to\infty} \omega_t(W(f)) = \omega_0(W(\Omega_-^* f)) := E_{\rm stat}(f)$$

and defines a quasi-free invariant state $\omega_{\rm stat}(f)$. Corresponding to the decomposition (3.7) of the initial state,

$$E_{\rm stat}(f) = \int \mathrm{d}\mu_1(\theta_1)\,\mathrm{d}\mu_2(\theta_2)\,E_{\rm stat}^{\theta_1,\theta_2}(f), \tag{3.9}$$

where $E_{\rm stat}^{\theta_1,\theta_2}(f) = E_0^{\theta_1,\theta_2}(\Omega_-^* f)$.

In view of the explicit forms of the functionals $E_0^{\theta_1,\theta_2}(\cdot)$, equation (3.8), and of the expression of Ω_-^* in terms of the (adjoint of the) kernel K, equation (2.19), proposition 3.1 provides a detailed description of the stationary state and allows the calculation of various quantities of physical interest. A few examples are given in the next section.

4. The particle and energy currents in $\omega_{\rm stat}$

We calculate here the particle and energy currents in the stationary states with the generating functionals $E_{\rm stat}^{\theta_1,\theta_2}$ entering the extremal decomposition (3.9). In doing this, we take advantage that the initial state, being a product of extremal equilibrium states, can be approximated by finite-volume states (possibly with weak symmetry-breaking perturbations), what allows us to substantiate the formal calculation below.

Let $\phi_s^{1,\Lambda}$ be the gauge automorphism on the sites in Λ of the first reservoir:

$$\phi_s^{1,\Lambda}(W(f))_{i,x} = \begin{cases} W(\mathrm{e}^{\mathrm{i}s} f_{1,x}), & \text{if} \quad i = 1, x \in \Lambda \\ W(f_{i,x}), & \text{otherwise.} \end{cases}$$

If Λ is finite, $\phi_s^{1,\Lambda}$ is the inner automorphism implemented by the unitary group $\mathrm{e}^{\mathrm{i}sN^{1,\Lambda}}$. If H_λ^Λ is the second quantized Hamiltonian (1.1) (with summations restricted to Λ), plus the small symmetry breaking term, the operator corresponding to the current of particles flowing from the first to the second reservoir at time t is given by the time derivative of the operator $N^{1,\Lambda}$, which is (up to terms of the order of the small symmetry breaking term)

$$I_{\rm part}^{1,\Lambda}(t) = \mathrm{e}^{\mathrm{i}tH_\lambda^\Lambda}\,\mathrm{i}\big[H_\lambda^\Lambda, N^{1,\Lambda}\big]\,\mathrm{e}^{-\mathrm{i}tH_\lambda^\Lambda} = -\mathrm{i}\lambda\,\mathrm{e}^{\mathrm{i}tH_\lambda^\Lambda}(a_0^* b_0 - b_0^* a_0)\,\mathrm{e}^{-\mathrm{i}tH_\lambda^\Lambda}. \tag{4.1}$$

Calculating the expectation in the finite-volume states approximating the initial state, letting $\Lambda \nearrow \mathbb{Z}^3$, followed by the suppression of the symmetry breaking field and, finally, $t \to \infty$, one obtains the following expression for the particle current in the stationary state:

$$J_{\rm part}^1 = -\mathrm{i}\lambda\omega_{\rm stat}\big(a^*(\delta_0^1)a(\delta_0^2) - a^*(\delta_0^2)a(\delta_0^1)\big) = 2\lambda\,\mathrm{Im}\,\big(\omega_{\rm stat}\big(a^*(\delta_0^1)a(\delta_0^2)\big)\big), \tag{4.2}$$

where $a^\sharp(f)$ denote the creation/annihilation operators in the Gelfand–Naimark–Segal representation [1] of the quasi-free state $\omega_{\rm stat}$; here, $\big(\delta_0^i\big)(x)_j = \delta_{i,j}\delta_{x,0}$.

Similar considerations for $H_0^{1,\Lambda} = \frac{1}{2}\sum_{x,y\in\Lambda} t_{x,y} a_x^* a_y$ instead of $N^{1,\Lambda}$ provide the expression for the energy current

$$J_{\rm en}^1 = -\mathrm{i}\frac{\lambda}{2}\omega_{\rm stat}\big(a^*(h_0^1)a(\delta_0^2) - a^*(\delta_0^2)a(h_0^1)\big) = \lambda\,\mathrm{Im}\,\big(\omega_{\rm stat}\big(a^*(h_0^1)a(\delta_0^2)\big)\big), \tag{4.3}$$

where

$$h_0^1(x)_j = -\delta_{j,1}(\delta_{|x|,1} - 6\delta_{x,0}).$$

As announced, we first consider the expression of the currents in the stationary state $\omega_{\mathrm{stat}}^{\theta_1,\theta_2}$ corresponding to extremal initial states, i.e., defined by the generating functional $E_{\mathrm{stat}}^{\theta_1,\theta_2}(f) = E_0^{\theta_1,\theta_2}(\Omega_-^* f)$, where $E_0^{\theta_1,\theta_2}$ is given by equation (3.8). The following formulae are obtained using the well-known expressions for the two-point functions in a quasi-free state [9]:

Lemma 4.1. *In a quasi-free state ω over $\mathcal{D}$ with generating functional*

$$\omega(W(f)) = \mathrm{e}^{-\frac{1}{4}(f,Xf)}\mathrm{e}^{\mathrm{i}\sqrt{2}\mathrm{Re}(\xi,f)},$$

where $X \geqslant 1$ is a self-adjoint operator in $\mathcal{H}$ with form domain $Q(X) \supset \mathcal{D}$, and $\xi \in \mathcal{D}'$),

$$\omega(a(f)) = \overline{(\xi, f)}\omega(a^*(g)a(f)) = (f, Xg) + \omega(a^*(g))\omega(a(f)).$$

Plugging this into equation (4.2), one gets the following expression of the particle current in the stationary state:

$$\begin{aligned} J_{\mathrm{part}}^1(\theta_1,\theta_2) &= 2\lambda\,\mathrm{Im}\,\omega_0^{\theta_1,\theta_2}\big(a_0^*\big(\Omega_-^*\big(\delta_0^1\big)\big)a_0\big(\Omega_-^*\big(\delta_0^2\big)\big)\big) \\ &= \frac{2\lambda}{(2\pi)^3}\int (n_2(k) - n_1(k))\frac{\mathrm{Im}\,\gamma(k)}{|1-\gamma(k)^2|^2}\,\mathrm{d}^3k \\ &\quad + \frac{2\lambda}{(2\pi)^3}\frac{\sqrt{\rho_{01}\rho_{02}}}{1-\gamma(0)^2}\sin(\theta_2-\theta_1). \end{aligned} \tag{4.4}$$

Likewise, one gets for the stationary energy current

$$\begin{aligned} J_{\mathrm{en}}^1(\theta_1,\theta_2) &= 2\lambda\,\mathrm{Im}\,\omega_0^{\theta_1,\theta_2}\big(a_0^*\big(\Omega_-^*\big(h_0^1\big)\big)a_0\big(\Omega_-^*\big(\delta_0^2\big)\big)\big) \\ &= \frac{2\lambda^2}{(2\pi)^3}\int (n_2(k) - n_1(k))\frac{\mathrm{Im}\,\gamma(k)}{|1-\gamma(k)^2|^2}\,\mathrm{d}^3k. \end{aligned} \tag{4.5}$$

5. Conclusion

We have shown that, for the solvable toy model defined by equation (1.1), an initial state, which is the product of extremal equilibrium states of the two reservoirs, reaches a stationary state with flow of particles and energy through the junction. Equations (4.4), (4.5) cover the whole range of temperatures, densities and phases of the initial equilibrium states of the two Bose gases. Several remarks are in order.

If both reservoirs are condensed, i.e., ρ_{01} and ρ_{02} are both different from zero, the particle current shows a peculiar dependence on the phase difference. This is not true for the energy current, where the second term coming from the expectations of the creation/annihilation operators does not contribute. Also, if $\rho_{01}\rho_{02} \neq 0$ and $\beta_1 = \beta_2$, then $n_1(k) = n_2(k)$, in which case the integral terms in equations (4.4), (4.5), representing the contribution to the currents of the excited states, vanish; therefore, particles are exchanged only between the $k = 0$ states and there is no energy flow (as expected, as the $k = 0$ states carry no energy).

In order to obtain the currents in the canonical state, we have still to integrate expressions (4.4), (4.5) over the phases θ_i of the two condensates. This has the effect that the particle currents between the $k = 0$ states are averaged out and only the first term in equation (4.4) survives. In particular, there is no current if the temperatures are equal and either $\rho_1 = \rho_2 \leqslant \rho_{\mathrm{cr}}(\beta)$ or both densities are overcritical (irrespective of their values).

In conclusion, the presence of the condensates in the reservoirs has little influence on the currents, as long as one considers non-symmetry-breaking states. We conjecture that this holds true for more general junctions and we propose to check this fact for a model like (1.1) with polynomial interaction terms of second degree in $a_0^\sharp, b_0^\sharp$.

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